THE CHINESE UNIVERSITY OF HONG KONG
DEPARTMENT OF MATHEMATICS

MMAT5220 Complex Analysis and its Applications 2016-2017
Suggested Solution to Assignment 6

1 We denote the function in each parts by $f(z)$.

(a) Since $z^2 - 5z + 4 = 0$ if and only if $z = 1$ or $z = 4$, 1 and 4 are two singular points of the function $f(z)$.

For $z = 1$, note that for $0 < |z - 1| < 1$,

$$f(z) = \frac{z-1}{z^2 - 5z + 4} = \frac{z-1}{(z-1)(z-4)} = \frac{1}{z-4}$$

Since $\dfrac{1}{z-4}$ is analytic for $0 \leq |z-1| < 1$, the principal part of $f(z)$ at $z = 1$ is 0 and $z = 1$ is a removable singularity for $f(z)$. Furthermore, the residue at $z = 1$ is given by 0.

For $z = 4$, note that for $0 < |z - 4| < 1$,

$$f(z) = \frac{1}{z-4}$$

Therefore, the principal part of $f(z)$ at $z = 4$ is given by $\dfrac{1}{z-4}$ and $z = 4$ is a simple pole of $f(z)$. Futhermore, the residue at $z = 4$ is given by 1.

(b) Note that 0 is a singular point of the function $f(z)$. For $0 < |z| < 1$, we have

$$f(z) = \sin\left(\frac{2}{z}\right) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{2}{z}\right)^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}\left(\frac{1}{z^{2n+1}}\right)$$

Therefore, the principal part of $f(z)$ at $z = 0$ is given by $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!}\left(\frac{1}{z^{2n+1}}\right)$ and $z = 0$ is an essential singularity of $f(z)$. Futhermore, the residue at $z = 0$ is given by 2.

(c) Since $\cos z = 0$ if and only if $z = N\pi + \dfrac{\pi}{2}$ for some $N \in \mathbb{Z}$, $N\pi + \dfrac{\pi}{2}$ are singular points of the function $f(z)$ for any $N \in \mathbb{Z}$.

For $0 < |z - \left(N\pi + \frac{\pi}{2}\right)| < 1$,

$$\begin{aligned}
f(z) =& \frac{z+1}{\cos z} \\
=& \frac{\left(z - (N\pi + \frac{\pi}{2})\right) + \left(N\pi + \frac{\pi}{2} + 1\right)}{(-1)^{N+1}\sin\left(z - (N\pi + \frac{\pi}{2})\right)} \\
=& (-1)^{N+1} \frac{\left(z - (N\pi + \frac{\pi}{2})\right) + \left(N\pi + \frac{\pi}{2} + 1\right)}{\left(z - (N\pi + \frac{\pi}{2})\right) - \frac{1}{3!}\left(z - (N\pi + \frac{\pi}{2})\right)^3 + \frac{1}{5!}\left(z - (N\pi + \frac{\pi}{2})\right)^5 + \dots} \\
=& (-1)^{N+1}\left[\left(z - (N\pi + \frac{\pi}{2})\right) + \left(N\pi + \frac{\pi}{2} + 1\right)\right]\left(\frac{1}{\left(z - (N\pi + \frac{\pi}{2})\right)} + \dots\right) \\
=& (-1)^{N+1}\frac{\left(N\pi + \frac{\pi}{2} + 1\right)}{z - \left(N\pi + \frac{\pi}{2}\right)} + \dots
\end{aligned}$$

As a result, for any $N \in \mathbb{Z}$, the principal part of $f(z)$ at $z = N\pi + \frac{\pi}{2}$ is given by

$$(-1)^{N+1}\frac{\left(N\pi + \frac{\pi}{2} + 1\right)}{z - (N\pi + \frac{\pi}{2})}$$

and the point is a simple pole. Furthermore, the residue at that point is given by

$$(-1)^{N+1}\left(N\pi + \frac{\pi}{2} + 1\right)$$

(d) It is clear that $z = 0$ is a singular point. Furthermore, for $0 < |z| < 1$,

$$f(z) = \frac{\sin 3z}{z} = \frac{1}{z}\sum_{n=0}^{\infty}\frac{(-1)^n(3z)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n 3^{2n+1}z^{2n}}{(2n+1)!}$$

Therefore, the principal part of $f(z)$ at $z = 0$ is given by 0 and $z = 0$ is an removable singularity of $f(z)$. Futhermore, the residue at $z = 0$ is given by 0.

(e) Note that since we are using the principal branch for the square root, $z = 4$ is the only singular point of $f(z)$. Furthermore, for $0 < |z - 4| < 1$,

$$f(z) = \frac{z^2}{2 - \sqrt{z}} = \frac{-z^2(2 + \sqrt{z})}{z - 4}$$

Let $\phi(z) = -z^2(2 + \sqrt{z})$. Note that $\phi(z)$ is analytic for $0 < |z - 4| < 1$. Moreover, we have

$$\phi(z) = \phi(4) + \phi'(4)(z - 4) + \frac{\phi''(4)}{2}(z - 4)^2 + \dots$$

Therefore,

$$f(z) = \frac{-z^2(2 + \sqrt{z})}{z - 4} = \frac{\phi(4)}{z - 4} + \phi'(4) + \frac{\phi''(4)}{2}(z - 4) + \dots$$

From this we can see that the principal part of $f(z)$ at $z = 4$ is given by $\dfrac{\phi(4)}{z-4} = \dfrac{-64}{z-4}$ and $z = 4$ is a simple pole of $f(z)$. Futhermore, the residue at $z = 4$ is given by -64.

2 In each part, we denote the integrand by $f(z)$.

(a) For $f(z) = \dfrac{2z - 3}{z(z+1)}$, note that the singular points $z = 0$ and $z = 1$ lie inside the contour $|z| = 3$. Moreover,

$$\frac{1}{z^2}f(\frac{1}{z}) = \frac{1}{z^2}\left(\frac{\frac{2}{z} - 3}{\frac{1}{z}(\frac{1}{z} + 1)}\right) = \frac{2 - 3z}{z(z+1)} = \frac{(2 - 3z)/(z+1)}{z}$$

As a result,

$$\int_{|z|=3}\frac{2z - 3}{z(z+1)}dz = 2\pi i \operatorname{Res}_{z=0}\frac{(2 - 3z)/(z+1)}{z} = 2\pi i\left(\frac{2 - 3(0)}{0 + 1}\right) = 4\pi i$$

(b) For $f(z) = \dfrac{z^3}{4 + z^2}$, note that the singular points $z = \pm 2i$ lie inside the contour $|z| = 3$. Moreover, for $0 < |z| < \frac{1}{2}$,

$$\frac{1}{z^2}f(\frac{1}{z}) = \frac{1}{z^2}\left(\frac{\frac{1}{z^3}}{4 + \frac{1}{z^2}}\right) = \frac{1}{z^3(4z^2+1)} = \frac{1}{z^3}(1 - 4z^2 + 16z^4 + \dots) = \frac{1}{z^3} - \frac{4}{z} + \dots,$$

As a result,

$$\int_{|z|=3}\frac{z^3}{4 + z^2} = 2\pi i \operatorname{Res}_{z=0}\frac{1}{z^3(4z^2+1)} = 2\pi i(-4) = -8\pi i$$

3 Since $q(z)$ is analytic and has a zero of order 1 at $z = z_0$, we have

$$q(z) = q'(z_0)(z-z_0) + \frac{q''(z_0)}{2}(z-z_0)^2 + \cdots = (z-z_0)\left(q'(z_0) + \frac{q''(z_0)}{2}(z-z_0) + \ldots\right) = (z-z_0)g(z),$$

where $g(z) = q'(z_0) + \frac{q''(z_0)}{2}(z - z_0) + \ldots$ is an analytic function near z_0 with $g(z_0) = q'(z_0) \neq 0$. As a result, since

$$f(z) = \frac{1}{[q(z)]^2} = \frac{1/[g(z)]^2}{(z-z_0)^2}$$

and the function $\phi(z) = \dfrac{1}{[g(z)]^2}$ is analytic near z_0 with $\phi(z_0) \neq 0$, z_0 is a pole of order 2. Furthermore,

$$\text{Res}_{z=z_0} f(z) = \phi'(z_0) = -2\frac{g'(z_0)}{[g(z_0)]^3} = -\frac{q''(z_0)}{[q'(z_0)]^3}$$

4 (a) Note that the singular points of the integrand $f(z) = \dfrac{1}{z^2 \sin(z)}$ inside the contour are given by $z = 0$ and $z = n\pi$ for $n = \pm 1, \pm 2, \ldots, \pm N$.

Note that for $0 < |z| < 1$,

$$f(z) = \frac{1}{z^2 \sin(z)} = \frac{1}{z^2\left(\sum_{k=0}^{\infty} \frac{(-1)^k z^{2k+1}}{(2k+1)!}\right)} = \frac{1}{z^3 - \frac{1}{6}z^5 + \frac{1}{120}z^7 + \ldots} = \frac{1}{z^3} + \frac{1}{6z} + \ldots$$

As a result, $\text{Res}_{z=0} f(z) = \dfrac{1}{6}$.

For $0 < |z - n\pi| < 1$, $n = \pm 1, \pm 2, \ldots, \pm N$,

$$\begin{aligned} f(z) =& \frac{1}{z^2 \sin(z)} \\ =& \frac{1/z^2}{(-1)^n \sin(z - n\pi)} \\ =& \frac{1/z^2}{(-1)^n (z - n\pi)[1 - \frac{1}{6}(z - n\pi) + \frac{1}{120}(z - n\pi)^2 + \ldots]} \\ =& \frac{(-1)^n/[z^2(1 - \frac{1}{6}(z - n\pi) + \frac{1}{120}(z - n\pi)^2 + \ldots)]}{z - n\pi} \\ =& \frac{\phi(z)}{z - n\pi}, \end{aligned}$$

where

$$\phi(z) = \frac{(-1)^n}{z^2[1 - \frac{1}{6}(z - n\pi) + \frac{1}{120}(z - n\pi)^2 + \ldots]}$$

is analyic near $z = n\pi$ with $\phi(n\pi) = \dfrac{(-1)^n}{n^2\pi^2}$.

As a result, $\text{Res}_{z=0} f(z) = \dfrac{(-1)^n}{n^2\pi^2}$ for any $n = \pm 1, \pm 2, \ldots, \pm N$. Therefore,

$$\int_{C_N} \frac{dz}{z^2 \sin z} = 2\pi i \sum_{n=-N}^{N} \text{Res}_{z=n\pi} \frac{1}{z^2 \sin z} = 2\pi i \left(\frac{1}{6} + 2\sum_{n=1}^{N} \frac{(-1)^n}{n^2\pi^2}\right)$$

(b) Recall the formula that for $z = x + iy$, we have

$$|\sin z|^2 = \sin^2 x + \sinh^2 y$$

In particular, on the contour $x = \pm \left(N + \frac{1}{2}\right)\pi$,

$$|\sin z|^2 \geq \sin^2 \left(N + \frac{1}{2}\right)\pi = 1$$

On the other hand, on the contour $y = \pm \left(N + \frac{1}{2}\right)\pi$,

$$|\sin z|^2 \geq \sinh^2 \left(N + \frac{1}{2}\right)\pi \geq \sinh^2 \left(\frac{\pi}{2}\right) \geq 1$$

Moreover, on the contour C_N, we have $|z| \geq \left(N + \frac{1}{2}\right)\pi$ Therefore,

$$\left|\int_{C_N} \frac{dz}{z^2 \sin z}\right| \leq \left[4\left(N + \frac{1}{2}\right)\pi\right] \frac{1}{\left(N + \frac{1}{2}\right)^2 \pi^2 (1)} = \frac{4}{\left(N + \frac{1}{2}\right)\pi} \xrightarrow{N \to \infty} 0$$

By a), we have

$$2\pi i \left(\frac{1}{6} + 2\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2 \pi^2}\right) = 0$$

which gives

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}.$$